Exercise 39

Find the work done by the force field

$$\mathbf{F}(x, y) = x\,\mathbf{i} + (y + 2)\,\mathbf{j}$$

in moving an object along an arch of the cycloid

$$\mathbf{r}(t) = (t - \sin t)\,\mathbf{i} + (1 - \cos t)\,\mathbf{j} \qquad 0 \le t \le 2\pi$$

Solution

Calculate the line integral of the force field over the cycloid to find the work done in moving the object.

$$\begin{aligned}
W &= \int_C \mathbf{F} \cdot d\mathbf{r} \\
&= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt \\
&= \int_0^{2\pi} \langle (t - \sin t), (1 - \cos t) + 2, 0 \rangle \cdot \langle 1 - \cos t, \sin t, 0 \rangle\,dt \\
&= \int_0^{2\pi} [(t - \sin t)(1 - \cos t) + (3 - \cos t)(\sin t) + (0)(0)]\,dt \\
&= \int_0^{2\pi} (t + 2\sin t - t\cos t)\,dt \\
&= \int_0^{2\pi} t\,dt + 2\int_0^{2\pi} \sin t\,dt - \int_0^{2\pi} t\cos t\,dt \\
&= \left(\frac{t^2}{2}\right)\Bigg|_0^{2\pi} + 2(-\cos t)\Big|_0^{2\pi} - \int_0^{2\pi} t\frac{d}{dt}(\sin t)\,dt \\
&= \frac{(2\pi)^2}{2} + 2(-1 + 1) - \Bigg[\underbrace{(t\sin t)\Big|_0^{2\pi}}_{=\,0} - \int_0^{2\pi} \frac{d}{dt}(t)\sin t\,dt\Bigg] \\
&= 2\pi^2 + \underbrace{\int_0^{2\pi} \sin t\,dt}_{=\,0} \\
&= 2\pi^2
\end{aligned}$$